Journal of Global Optimization **8**: 289–294, 1996.

Solving Two–Level Variational Inequality*

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Received January 22, 1994; Revised November 15, 1995

Abstract. An approach to solving a mathematical program with variational inequality or nonlinear complementarity constraints is presented. It consists in a variational re-formulation of the optimization criterion and looking for a solution of thus obtained variational inequality among the points satisfying the initial variational constraints.

Keywords: Variational inequality, parametrization, pseudo-monotone mapping, penalty function algorithm

1. Introduction

The problem of solving a mathematical program with variational inequalities or complementarity conditions as constraints arises quite frequently in the analysis of physical and socio-economic systems. According to a remark in the recent paper by P.T. Harker and S.-C. Choi [1], the current state-of-the-art for solving such problems is heuristic. The latter paper [1] presents an exterior-point penalty method based on M.J. Smith's optimization formulation of the finite-dimensional variational inequality problem [2]. Recently, we have learned of the paper by J. Outrata [3], in which attention is also paid to this type of optimization problems.

Another approach to solving the above-mentioned problem consists, on the contrary, in a variational re-formulation of the optimization criterion and looking for a solution of thus obtained variational inequality among the points satisfying the initial variational inequality constraints. In Section 2 of our paper, we examine conditions under which the set of the feasible points is non-empty, and compare the conditions with those established previously [6]. Section 3 describes a penalty function method solving the two-level problem after having reduced it to a single variational inequality with a penalty parameter.

2. Existence Theorem

Let X be a non-empty, closed, convex subset of R^n and G a continuous mapping from X into R^n. Suppose that G is pseudo-monotone with respect to X, i.e.

* This research was supported by the Russian Fundamental Research Foundation, Grant No. 93-012-842

$$(x-y)^{\mathrm{T}}G(y) \geq 0 \quad \text{implies} \quad (x-y)^{\mathrm{T}}G(x) \geq 0 \quad \forall x, y \in X, \tag{1}$$

and that there exists a vector $x^0 \in X$ such that

$$G(x^0) \in \text{int}\,(0^+X)^*, \tag{2}$$

where $\text{int}(\cdot)$ denotes the interior of the set. Here 0^+X is the recession cone of the set X, i.e. the set of all directions $s \in R^n$ such that $X + s \subset X$; at last, C^* is the dual cone of $C \subset R^n$, i.e.

$$C^* = \{y \in R^n : y^{\mathrm{T}}x \geq 0 \qquad \forall x \in C\}$$

Hence, condition (2) implies that the vector $G(x^0)$ lies within the interior of the dual to the recession cone of the set X.

Under these assumptions, the following result obtains:

PROPOSITION 1 *The variational inequality problem: to find a vector $z \in X$ such that*

$$(x-z)^{\mathrm{T}}G(z) \geq 0 \qquad \forall x \in X, \tag{3}$$

has a non-empty, compact, convex solution set.

Proof: It is well-known [4] that the pseudo-monotonicity (1) and continuity of the mapping G imply convexity of the problem (3) solution set

$$Z = \{z \in X : (x-z)^{\mathrm{T}}G(z) \geq 0 \qquad \forall x \in X\}, \tag{4}$$

if the latter is non-empty. Now we show the existence of at least one solution to this problem. In order to do that, we use the following fact [5]: if there exists a non-empty bounded subset D of X such that for every $x \in X \backslash D$ there is a $y \in D$ with

$$(x-y)^{\mathrm{T}}G(x) > 0, \tag{5}$$

then problem (3) has a solution. Moreover, the solution set (4) is bounded because $Z \subset D$. Now, we construct the set D as follows:

$$D = \{\, x \in X\colon\ (x-x^0)^{\mathrm{T}}G(x^0) \leq 0 \,\}. \tag{6}$$

The set D is clearly non-empty, since it contains the point x^0. Now we show that D is bounded, even if X is not so. On the contrary, suppose that a sequence $\{x^k\} \subseteq D$ is norm divergent, i.e. $\|x^k - x^0\| \to +\infty$ when $k \to \infty$. Without lack of generality, assume that $x^k \neq x^0,\ k = 1, 2, \ldots,$ and consider the inequality

$$\frac{(x^k - x^0)^{\mathrm{T}}G(x^0)}{\|x^k - x^0\|} \leq 0, \qquad k = 1, 2, \ldots, \tag{7}$$

which follows from definition (6) of the set D. Again not affecting generality, accept that the normed sequence $(x^k - x^0)/\|x^k - x^0\|$ converges to a vector $s \in R^n$, $\|s\| = 1$. It is well-known (cf. [6], Theorem 8.2) that $s \in 0^+X$. From (7), we deduce the limit relationship

$$s^{\mathrm{T}}G(x^0) \leq 0. \tag{8}$$

Since $0^+X \neq \{0\}$ (as X is unbounded and convex), we have $0 \in \text{fr } (0^+X)^*$, hence $G(x^0) \neq 0$. Now it is easy to see that inequality (8) contradicts assumption (2). Indeed, the inclusion $G(x^0) \in \text{int } (0^+X)^*$ implies that $s^{\mathrm{T}}G(x^0) > 0$ for any $s \in 0^+X, s \neq 0$. The contradiction proves boundedness of the set D, and the statement of Proposition 1 therewith. Really, for a given $x \in X \backslash D$, one can pick $y = x^0 \in D$ with the inequality $(x - y)^{\mathrm{T}}G(y) > 0$ taking place. The latter, jointly with the pseudo-monotonicity of G, implies the required condition (5). This completes the proof. ■

Remark. The assertion of Proposition 1 has been obtained earlier [7] under the same assumptions except for inclusion (2), which is obviously invariant with respect to an arbitrary translation of the set X followed by the corresponding transformation of the mapping G. Instead of (2), the authors [7] used another assumption $G(x^0) \in \text{int}(X^*)$ which is clearly not translation-invariant.

Now suppose further that the problem (3) solution set Z contains more than one element, and consider the following variational inequality problem: to find a vector $z^* \in Z$ such that

$$(z - z^*)^{\mathrm{T}}F(z^*) \geq 0 \qquad \text{for all} \quad z \in Z. \tag{9}$$

Here, the mapping $F: X \to R^n$ is continuous and strictly monotone over X; i.e.

$$(x - y)^{\mathrm{T}}[F(x) - F(y)] > 0 \qquad \forall x, y \in X, x \neq y.$$

In this case, the compactness and convexity of the set Z guarantee [5] existence of a unique (due to the strict monotonicity of F) solution z of problem (9). We refer to problem (3), (4), (9) as the two-level variational inequality (TLVI). In the next section, we present a penalty function algorithm solving the TLVI without explicit description of the set Z.

3. Penalty Function Method

Fix a positive parameter ε and consider the following parametric variational inequality problem: to find a vector $x_\varepsilon \in X$ such that

$$(x - x_\varepsilon)^{\mathrm{T}}[G(x_\varepsilon) + \varepsilon F(x_\varepsilon)] \geq 0 \qquad \text{for all} \quad x \in X. \tag{10}$$

If we assume that the mapping G is monotone over X, i.e.

$$(x-y)^{\mathrm{T}}[G(x)-G(y)] \geq 0 \qquad \forall x, y \in X, \tag{11}$$

and keep intact all the above assumptions regarding G, F and Z , then the following result obtains:

PROPOSITION 2 *For each sufficiently small value $\varepsilon > 0$, problem (10) has a unique solution x_ε. Moreover, x_ε converge to the solution z^* of TLVI (3), (4), (9) when* $\varepsilon \to 0$.

Proof: Since G is monotone and F is strictly monotone, the mapping $\Phi_\varepsilon = G+\varepsilon F$ is strictly monotone on X for any $\varepsilon > 0$. It is also clear that if x^0 satisfies (2) then the following inclusion holds

$$\Phi_\varepsilon(x^0) = G(x^0) + \varepsilon F(x^0) \in \text{int } (0^+X)^*, \tag{12}$$

if $\varepsilon > 0$ is small enough. Hence, Proposition 1 implies validity of the first assertion of Proposition 2; namely, for every $\varepsilon > 0$ satisfying (12), variational inequality (10) has a unique solution x_ε .

From the continuity of F and G, it follows that each (finite) limit point $\bar{x}$ of the generalized sequence $Q = \{x_\varepsilon\}$ of solutions to problem (10) solves variational inequality (3); that is, $\bar{x} \in Z$. Now we prove that the point $\bar{x}$ solves problem (9), too. In order to do that, we use the following relationships valid for any $z \in Z$ due to (4), (10) and (11):

$$(z-x_\varepsilon)^{\mathrm{T}}[G(z)-G(x_\varepsilon)] \geq 0, \tag{13}$$

$$(z-x_\varepsilon)^{\mathrm{T}}G(z) \leq 0, \tag{14}$$

$$(z-x_\varepsilon)^{\mathrm{T}}G(x_\varepsilon) \geq -\varepsilon(z-x_\varepsilon)^{\mathrm{T}}F(x_\varepsilon). \tag{15}$$

Subtracting (15) from (14) and using (13), we obtain the following series of inequalities

$$0 \leq (z-x_\varepsilon)^{\mathrm{T}}[G(z)-G(x_\varepsilon)] \leq \varepsilon(z-x_\varepsilon)^{\mathrm{T}}F(x_\varepsilon). \tag{16}$$

From (16) we have $(z-x_\varepsilon)^{\mathrm{T}}F(x_\varepsilon) \geq 0$ for all $\varepsilon > 0$ and $z \in Z$. Since F is continuous, the following limit relationship holds: $(z-\bar{x})^{\mathrm{T}}F(\bar{x}) \geq 0$ for each $z \in Z$, which means that $\bar{x}$ solves (9).

Thus we have proved that every limit point of the generalized sequence Q solves TLVI (3), (4), (9). Hence, Q can have at most one limit point. To complete proving Proposition 2, it suffices to establish that the set Q is bounded, and consequently, the limit point exists. In order to do that, consider a norm-divergent sequence $\{x_{\varepsilon_k}\}$ of solutions to parametric problem (10) where $\varepsilon_k \to 0$ as $k \to \infty$. Without loss the generality, suppose that $x_{\varepsilon_k} \neq x^0$ for each k, and $\dfrac{(x_{\varepsilon_k}-x^0)}{\|x_{\varepsilon_k}-x^0\|} \to s \in R^n$,

$\|s\| = 1$; here x^0 is the vector from condition (2). Since $\|x_{\varepsilon_k} - x^0\| \to +\infty$, we get $s \in 0^+X$ (cf. [6]). As the mappings G and F are monotone, the following inequalities take place for all $k = 1, 2, \ldots$:

$$(x_{\varepsilon_k} - x^0)^{\mathrm{T}}[G(x_{\varepsilon_k}) + \varepsilon_k F(x_{\varepsilon_k})] \le 0,$$

$$(x_{\varepsilon_k} - x^0)^{\mathrm{T}}[G(x^0) + \varepsilon_k F(x^0)] < 0. \tag{17}$$

Dividing inequality (17) by $\|x_{\varepsilon_k} - x^0\|$ we obtain

$$\frac{(x_{\varepsilon_k} - x^0)^{\mathrm{T}}}{\|x_{\varepsilon_k} - x^0\|} \cdot [G(x^0) + \varepsilon_k F(x^0)] \le 0, \qquad k = 1, 2, \ldots, \tag{18}$$

which implies (as $\varepsilon_k \to 0$) the limit inequality $s^{\mathrm{T}} G(x^0) \le 0$. Since $s \ne 0$, the latter inequality contradicts assumption (2). This contradiction demonstrates the set Q to be bounded which completes the proof of Proposition 2. ■

4. An Example

Let $\Omega \subseteq R^m$, $\Lambda \subseteq R^n$ be subsets of finite-dimensional Euclidean spaces and $f : \Omega \times \Lambda \to R$, $g : \Omega \times \Lambda \to R^n$ be continuous mappings. Consider the following mathematical program with variational inequality constraint:

$$\min_{(u,v) \in \Omega \times \Lambda} f(u, v), \tag{19}$$

s.t.

$$g(u, v)^T (w - v) \ge 0, \qquad \forall w \in \Lambda. \tag{20}$$

If function f is continuously differentiable, then problem (18)-(19) is obviously tantamount to TLVI (3), (4), (9) with the gradient mapping $f'(z)$ used as $F(z)$ and $G(u, v) = [0; g(u, v)]$; here $z = (u, v) \in \Omega \times \Lambda$.

As an example, examine the case when

$$f(u, v) = (u - v - 1)^2 + (v - 2)^2; \quad g(u, v) = uv; \quad \Omega = \Lambda = R^1_+.$$

Then it is readily verified that $z^* = (1; 0)$ and

$$\Phi_\varepsilon(u, v) = \big[\varepsilon(2u - 2v - 2); uv + \varepsilon(-2u + 4v - 2)\big].$$

Now solving the variational inequality: find $(u_\varepsilon, v_\varepsilon) \in R^2_+$ such that

$$\Phi_\varepsilon(u_\varepsilon, v_\varepsilon)^T \big[(u, v) - (u_\varepsilon, v_\varepsilon)\big] \ge 0 \quad \forall (u, v) \in R^2_+,$$

we obtain

$$u_\varepsilon = v_\varepsilon + 1; \quad v_\varepsilon = -\frac{1}{2} - \varepsilon + \sqrt{(\frac{1}{2} + \varepsilon)^2 + 4\varepsilon}.$$

Clearly $(u_\varepsilon, v_\varepsilon) \to z^*$ when $\varepsilon \to 0$.

Unfortunately, it is not always the case, because the mapping $[0; g(u, v)]$ is not usually monotone with respect to (u, v), even if $g(u, v)$ is such with respect to v for each fixed u.

Acknowledgments

The authors would like to thank the referees for their careful review of the manuscript and for the helpful comments and constructive suggestions that they provided.

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